# The Derivation algebra and automorphism group of the generalized Ramond $N = 2$ superconformal algebra [*]

**Jiayuan Fu**    **Yongcun Gao**

*Department of Mathematics, Communication University of China,*

*Beijing 100024, China*

*Email: fujy@cuc.edu.cn*

## Abstract

In this paper, we give the definition of the generalized Ramond $N = 2$ superconformal algebras and discuss the derivation algebra and the automorphism group.

## 1. Introduction

Superconformal algebras have been constructed for almost three decades([1, 9]). Since then the study of superconformal algebras has made much progress in both mathematics and physics. Kac and van de Leuer ([10]), Cheng and Kac ([3]) have classified all possible superconformal algebras and then Kac has proved that their classification is complete.

Structure theory and representation theory are also the most important two theories on Lie superalgebras. For $N = 2$ superconformal algebras, remarkable efforts have been taken by several research groups ([2, 4, 5, 6, 12, 11]). In [7], the modules of intermediate series over Ramond $N = 2$ superconformal algebra has been classified. For the Ramond $N = 2$ superconformal algebra, its even part is the so called twisted Heisenberg-Virasoro algebra ([13]). The derivation algebras of infinite-dimensional Heisenberg algebra, Virasoro algebra and the twisted Heisenberg-Virasoro algebra are given by Jiang([8]), Zhu, Meng([18]) and Shen, Jiang ([13]) respectively. Su, Song and Zhou have given the structures of derivation algebra of Weyl and Block type algebras in [14, 15, 16, 17]. Most of those algebras are finitely generated. In this paper, we will give the derivation algebra and the automorphism group of the generalized Ramond $N = 2$ superconformal algebra. In general, this algebra is infinitely generated.

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Let $\mathbb{F}$ be a field of characteristic zero, $\Gamma$ be an additive subgroup of $\mathbb{F}$, $\mathbf{0}$ be the identity element of $\Gamma$. The generalized Ramond $N = 2$ algebra is defined as below:

$$\mathcal{L} = \mathcal{L}_{\overline{0}} \oplus \mathcal{L}_{\overline{1}},$$

where

$$\mathcal{L}_{\overline{0}} = \operatorname{span}_{\mathbb{C}}\{L_\alpha, H_\beta, c \mid \alpha, \beta \in \Gamma\}, \quad \mathcal{L}_{\overline{1}} = \operatorname{span}_{\mathbb{C}}\{G_\alpha^-, G_\beta^+ \mid \alpha, \beta \in \Gamma\},$$

with the following relations:

$$
\begin{aligned}
{[L_\alpha, L_\beta]} &= (\alpha - \beta)L_{\alpha+\beta} + \tfrac{c}{12}(\alpha^3 - \alpha)\delta_{\alpha+\beta,0}, \\
{[H_\alpha, H_\beta]} &= \tfrac{\alpha}{3}c\delta_{\alpha+\beta,0}, \\
{[L_\alpha, H_\beta]} &= -\beta H_{\alpha+\beta}, \\
{[L_\alpha, G_\beta^\pm]} &= (\tfrac{\alpha}{2} - \beta)G_{\alpha+\beta}^\pm, \\
{[H_\alpha, G_\beta^\pm]} &= \pm G_{\alpha+\beta}^\pm, \\
{[G_\alpha^-, G_\beta^+]} &= 2L_{\alpha+\beta} - (\alpha - \beta)H_{\alpha+\beta} + \tfrac{c}{3}(\alpha^2 - \tfrac{1}{4})\delta_{\alpha+\beta,0},
\end{aligned}
\tag{1.1}
$$

where $\delta_{m,n}$ is the Kronecker notation which satisfies that $\delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$.

Throughout this paper, $\mathcal{L}$ denotes the **generalized Ramond $N = 2$ superconformal algebra**. It is easy to see that $\mathcal{L}$ is a $\Gamma$-graded algebra:

$$\mathcal{L} = \bigoplus_{\alpha \in \Gamma} \mathcal{L}_\alpha,$$

where $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [L_0, x] = -\alpha x\} = \operatorname{span}_{\mathbb{F}}\{L_\alpha, H_\alpha, G_\alpha^\pm, \delta_{\alpha,0}c\}$.

In the following section, we will discuss the derivation algebra of $\mathcal{L}$. Our main result in this section is theorem 2.3. Then the automorphism group will be given in the last section, and we will obtain the main theorem 3.4.

## 2. The derivation algebra of $\mathcal{L}$

In this section, we denote the derivation algebra of $\mathcal{L}$ by $Der\mathcal{L}$, the set of inner derivations by $ad\mathcal{L}$. Since $\mathcal{L}$ is a $\mathbb{Z}_2$-graded algebra, then

$$Der\mathcal{L} = (Der\mathcal{L})_{\overline{0}} \oplus (Der\mathcal{L})_{\overline{1}},$$

where

$$(Der\mathcal{L})_{\overline{0}} = \{D \in Der\mathcal{L} \mid D(\mathcal{L}_{\overline{i}}) \subseteq \mathcal{L}_{\overline{i}}, \ \overline{i} \in \mathbb{Z}_2\},$$

denotes the set of even derivations of $\mathcal{L}$, and

$$(Der\mathcal{L})_{\overline{1}} = \{D \in Der\mathcal{L} \mid D(\mathcal{L}_{\overline{i}}) \subseteq \mathcal{L}_{\overline{i+1}}, \ \overline{i} \in \mathbb{Z}_2\},$$

denotes the set of odd derivations of $\mathcal{L}$. Note that $\mathcal{L}$ is $\Gamma$-graded, we have that $Der\mathcal{L}$ is also $\Gamma$-graded: define

$$(Der\mathcal{L})_\gamma = \{D \in Der\mathcal{L} \mid D(\mathcal{L}_\beta) \subseteq \mathcal{L}_{\beta+\gamma}, \beta \in \Gamma\}.$$

Then we have that $Der\mathcal{L} = \underset{\gamma \in \Gamma}{\oplus} (Der\mathcal{L})_\gamma$ (see [15, 16, 14, 17]). Obviously, the $\Gamma$-graded and $\mathbb{Z}_2$-graded structures are compatible, i.e., $Der\mathcal{L}_{\bar{0}} = \underset{\gamma \in \Gamma}{\oplus} (Der\mathcal{L})'_\gamma$, $Der\mathcal{L}_{\bar{1}} = \underset{\gamma \in \Gamma}{\oplus} (Der\mathcal{L})''_\gamma$, where $(Der\mathcal{L})'_\gamma \oplus (Der\mathcal{L})''_\gamma = (Der\mathcal{L})_\gamma$, the $\mathbb{Z}_2$-graded structure of $(Der\mathcal{L})_\gamma$ is similar. In the following, we will only discuss the homogenous derivation $D$.

**Lemma 2.1.** The odd derivations of $\mathcal{L}$ are all inner derivations, i.e., $(Der\mathcal{L})_{\bar{1}} = ad\mathcal{L}_{\bar{1}}$.

**Proof.** Obviously, we only need to prove that $(Der\mathcal{L})_{\bar{1}} \subseteq ad\mathcal{L}_{\bar{1}}$. If $D \in (Der\mathcal{L})_{\bar{1}} \cap (Der\mathcal{L})_\gamma$, $\gamma \in \Gamma$, suppose that

$$
\begin{aligned}
D(L_\alpha) &= a_{\gamma,\alpha} G^+_{\alpha+\gamma} + b_{\gamma,\alpha} G^-_{\alpha+\gamma}, \\
D(H_\alpha) &= c_{\gamma,\alpha} G^+_{\alpha+\gamma} + d_{\gamma,\alpha} G^-_{\alpha+\gamma}, \\
D(G^+_\alpha) &= e_{\gamma,\alpha} L_{\alpha+\gamma} + f_{\gamma,\alpha} H_{\alpha+\gamma} + m_\alpha \delta_{\alpha+\gamma,0} c, \\
D(G^-_\alpha) &= g_{\gamma,\alpha} L_{\alpha+\gamma} + h_{\gamma,\alpha} H_{\alpha+\gamma} + n_\alpha \delta_{\alpha+\gamma,0} c.
\end{aligned}
$$

Since $D$ is a derivation, by the following equations:

$$
\begin{aligned}
D[H_\alpha, H_\beta] &= [D(H_\alpha), H_\beta] + [H_\alpha, D(H_\beta)], \\
D[L_\alpha, H_\beta] &= [D(L_\alpha), H_\beta] + [L_\alpha, D(H_\beta)], \qquad (2.1)\\
D[H_\alpha, G^\pm_\beta] &= [D(H_\alpha), G^\pm_\beta] + [H_\alpha, D(G^\pm_\beta)],
\end{aligned}
$$

for all $\alpha, \beta \in \Gamma$, we can obtain some relations between these coefficients:

$$
\begin{aligned}
c_{\gamma,\alpha} &= c_{\gamma,\beta}, \\
d_{\gamma,\alpha} &= d_{\gamma,\beta}, \\
e_{\gamma,\alpha+\beta} &= 2d_{\gamma,\alpha}, \\
f_{\gamma,\alpha+\beta} &= \alpha e_{\gamma,\beta} - (\alpha + \gamma - \beta) d_{\gamma,\alpha}, \\
m_{\alpha+\beta} &= \tfrac{1}{3}(\beta^2 - \tfrac{1}{4}) d_{\gamma,\alpha} + \tfrac{\alpha}{3} f_{\gamma,\beta}, \quad \alpha + \beta + \gamma = 0, \\
g_{\gamma,\alpha+\beta} &= -2c_{\gamma,\alpha}, \\
h_{\gamma,\alpha+\beta} &= -\alpha g_{\gamma,\beta} - (\alpha + \gamma - \beta) c_{\gamma,\alpha}, \\
n_{\alpha+\beta} &= -\tfrac{1}{3}(\beta^2 - \tfrac{1}{4}) c_{\gamma,\alpha} - \tfrac{\alpha}{3} h_{\gamma,\beta}, \quad \alpha + \beta + \gamma = 0, \\
\beta c_{\gamma,\alpha+\beta} &= a_{\gamma,\alpha} - (\tfrac{\alpha}{2} - \beta - \gamma) c_{\gamma,\beta}, \\
\beta d_{\gamma,\alpha+\beta} &= -b_{\gamma,\alpha} - (\tfrac{\alpha}{2} - \beta - \gamma) d_{\gamma,\beta}.
\end{aligned}
$$

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It is not difficult to see that $c_{\gamma,\alpha}$ and $d_{\gamma,\alpha}$ are contents. Set $c_{\gamma,\alpha} = \xi_0, d_{\gamma,\alpha} = \xi_1$, for all $\alpha \in \Gamma$, where $\xi_0, \xi_1 \in \mathbb{F}$, then we can obtain the following results:

$$
\begin{aligned}
a_{\gamma,\alpha} &= (\tfrac{\alpha}{2} - \gamma)\xi_0, & b_{\gamma,\alpha} &= -(\tfrac{\alpha}{2} - \gamma)\xi_1, \\
e_{\gamma,\alpha} &= 2\xi_1, & f_{\gamma,\alpha} &= (\alpha - \gamma)\xi_1, \\
g_{\gamma,\alpha} &= -2\xi_0, & h_{\gamma,\alpha} &= (\alpha - \gamma)\xi_0, \\
m_\alpha &= \tfrac{1}{3}(\alpha^2 - \tfrac{1}{4})\xi_1, & n_\alpha &= -\tfrac{1}{3}(\alpha^2 - \tfrac{1}{4})\xi_0.
\end{aligned}
$$

By these equations, we can easily check that $D = ad(-\xi_0 G_\gamma^+ + \xi_1 G_\gamma^-)$. $\qquad\square$

**Lemma 2.2.** If $D \in (Der\mathcal{L})_{\bar{0}} \cap (Der\mathcal{L})_\gamma$, and $\gamma \neq \mathbf{0}$, then $D \in ad\mathcal{L}$.

**Proof.** $D(c) = 0$, since $\gamma \neq \mathbf{0}$. Without confusion, we use the same symbols: suppose that

$$
\begin{aligned}
D(L_\alpha) &= a_{\gamma,\alpha}L_{\alpha+\gamma} + b_{\gamma,\alpha}H_{\alpha+\gamma} + m_\alpha\delta_{\alpha+\gamma,0}c, \\
D(H_\alpha) &= c_{\gamma,\alpha}L_{\alpha+\gamma} + d_{\gamma,\alpha}H_{\alpha+\gamma} + n_\alpha\delta_{\alpha+\gamma,0}c, \\
D(G_\alpha^+) &= e_{\gamma,\alpha}G_{\alpha+\gamma}^+ + f_{\gamma,\alpha}G_{\alpha+\gamma}^-, \\
D(G_\alpha^-) &= g_{\gamma,\alpha}G_{\alpha+\gamma}^+ + h_{\gamma,\alpha}G_{\alpha+\gamma}^-.
\end{aligned}
$$

Also by (2.1) and

$$
D[L_\alpha, G_\beta^\pm] = [D(L_\alpha), G_\beta^\pm] + [L_\alpha, D(G_\beta^\pm)], \tag{2.2}
$$

$$
D[G_\alpha^-, G_\beta^+] = [D(G_\alpha^-), G_\beta^+] + [G_\alpha^-, D(G_\beta^+)], \tag{2.3}
$$

we have that

$$
\begin{aligned}
\beta c_{\gamma,\alpha} &= \alpha c_{\gamma,\beta}, \\
e_{\gamma,\alpha+\beta} &= (\frac{\alpha+\gamma}{2} - \beta)c_{\gamma,\alpha} + d_{\gamma,\alpha} + e_{\gamma,\beta}, \\
f_{\gamma,\alpha+\beta} &= -f_{\gamma,\beta}, \qquad g_{\gamma,\alpha+\beta} = -g_{\gamma,\beta}, \\
h_{\gamma,\alpha+\beta} &= -(\frac{\alpha+\gamma}{2} - \beta)c_{\gamma,\alpha} + d_{\gamma,\alpha} + h_{\gamma,\beta}, \\
-\beta c_{\gamma,\alpha+\beta} &= (\alpha - \beta - \gamma)c_{\gamma,\beta} \\
(\frac{\alpha}{2} - \beta)e_{\gamma,\alpha+\beta} &= (\frac{\alpha+\gamma}{2} - \beta)a_{\gamma,\alpha} + b_{\gamma,\alpha} + (\frac{\alpha}{2} - \beta - \gamma)e_{\gamma,\beta}, \\
(\frac{\alpha}{2} - \beta)h_{\gamma,\alpha+\beta} &= (\frac{\alpha+\gamma}{2} - \beta)a_{\gamma,\alpha} - b_{\gamma,\alpha} + (\frac{\alpha}{2} - \beta - \gamma)h_{\gamma,\beta}, \\
-\beta n_{\alpha+\beta} &= \frac{\alpha+\gamma}{3}b_{\gamma,\alpha} + \frac{\alpha^3 - \alpha}{12}c_{\gamma,\beta}, \quad \alpha + \beta + \gamma = 0, \\
2a_{\gamma,\alpha+\beta} - (\alpha - \beta)c_{\gamma,\alpha+\beta} &= 2h_{\gamma,\alpha} + 2e_{\gamma,\beta}, \\
2b_{\gamma,\alpha+\beta} - (\alpha - \beta)d_{\gamma,\alpha+\beta} &= -(\alpha + \gamma - \beta)h_{\gamma,\alpha} - (\alpha - \beta - \gamma)e_{\gamma,\beta}, \\
6m_{\alpha+\beta} - 3(\alpha - \beta)n_{\alpha+\beta} &= (\beta^2 - \frac{1}{4})h_{\gamma,\alpha} + (\alpha^2 - \frac{1}{4})e_{\gamma,\beta}, \alpha + \beta = -\gamma.
\end{aligned}
$$

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From these equations, we can obtain that for all $\alpha \in \Gamma$

$$c_{\gamma,\alpha} = f_{\gamma,\alpha} = g_{\gamma,\alpha} = 0, \; b_{\gamma,\alpha} = \eta_\gamma, \; n_\alpha = \frac{1}{3}\eta_\gamma,$$

where $\eta_\gamma \in \mathbb{F}$, and

$$
\begin{aligned}
a_{\gamma,\alpha} &= \gamma^{-1}(\gamma - \alpha)(2h_{\gamma,0} + 2\gamma^{-1}\eta_\gamma), & d_{\gamma,\alpha} &= -\gamma^{-1}\alpha(2h_{\gamma,0} + 2\gamma^{-1}\eta_\gamma), \\
e_{\gamma,\alpha} &= \gamma^{-1}(\gamma - 2\alpha)h_{\gamma,0} + 2\gamma^{-2}(\gamma - \alpha)\eta_\gamma, & h_{\gamma,\alpha} &= \gamma^{-1}(\gamma - 2\alpha)h_{\gamma,0} - 2\gamma^{-2}\alpha\eta_\gamma, \\
m_\alpha &= \tfrac{1}{6}\gamma^{-1}(\alpha - \alpha^3)(h_{\gamma,0} + \gamma^{-1}\eta_\gamma). &
\end{aligned}
$$

It is not difficult to see that, if $D \in (Der\mathcal{L})_{\bar{0}} \cap (Der\mathcal{L})_\gamma$, and $\gamma \neq 0$,

$$D = ad(k_1 L_\gamma + k_2 H_\gamma),$$

where $k_1 = 2\gamma^{-1}(h_0 + \gamma^{-1}\eta_\gamma), k_2 = \gamma^{-1}\eta_\gamma, h_0, \eta_\gamma \in \mathbb{F}$. $\qquad\square$

If $D \in (Der\mathcal{L})_{\bar{0}} \cap (Der\mathcal{L})_0$, the case is a little difference, and we should note that $D(c) = i_0 c, i_0 \in \mathbb{F}$. By (2.1)-(2.3), used the similar method, we can obtain that for all $\alpha, \beta \in \Gamma$.

$$n_{\gamma,\alpha} = b_{\gamma,\alpha} = c_{\gamma,\alpha} = f_{\gamma,\alpha} = g_{\gamma,\alpha} = i_0 = m_0 = 0,$$

and

$$
\begin{cases}
h_{\gamma,\alpha} = e_\alpha - 2e_0, \quad a_{\gamma,\alpha} = d_{\gamma,\alpha} = e_\alpha - e_0, \\
a_{\gamma,\alpha+\beta} = a_{\gamma,\alpha} + a_{\gamma,\beta}.
\end{cases}
$$

Denote by $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$ the set of additive group homomorphisms from $\Gamma$ to $\mathbb{F}$. For $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, we define a derivation also denoted $\varphi$ by $\varphi(x_\alpha) = \varphi(\alpha)x_\alpha$, for all $x_\alpha \in \mathcal{L}$. We have the following theorem:

**Theorem 2.3.** $Der\mathcal{L}$ is spanned by $ad\mathcal{L}$ and $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$. In details, $Der\mathcal{L} = \bigoplus_{\alpha \in \Gamma}(Der\mathcal{L})_\alpha = (Der\mathcal{L})_{\bar{0}} \oplus (Der\mathcal{L})_{\bar{1}}$, where

$$
(Der\mathcal{L})_{\bar{i}} =
\begin{cases}
ad\mathcal{L}_{\bar{1}}, & \bar{i} = \bar{1}, \\
ad\mathcal{L}_\alpha, & \bar{i} = \bar{0}, \alpha \neq 0, \\
ad\mathcal{L}'_0 + \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F}), & \bar{i} = \bar{0}, \alpha = 0,
\end{cases}
$$

where $\mathcal{L}'_0 = \mathrm{span}_{\mathbb{F}}\{L_0, H_0\}$, and $Der\mathcal{L}_{\bar{0}} \cap ad\mathcal{L}_0 = \{ad(kL_0 + lH_0) \mid k, l \in \mathbb{F}\}$.

**Proof.** By the above argument, we know that if $D \in (Der\mathcal{L})_{\bar{0}} \cap (Der\mathcal{L})_0$, then

$$
\begin{aligned}
D(L_\alpha) &= \varphi(\alpha)L_\alpha, & D(H_\alpha) &= \varphi(\alpha)H_\alpha, & D(G_\alpha^+) &= (\varphi(\alpha) + e_0)G_\alpha^+, \\
D(G_\alpha^-) &= (\varphi(\alpha) - e_0)G_\alpha^-, & D(c) &= 0,
\end{aligned}
$$

where $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, i.e. $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$ for any $\alpha, \beta \in \Gamma$, and $e_0 \in \mathbb{F}$. Obviously, $D \in ad\mathcal{L}$ if and only if $\varphi(\alpha) = k\alpha, k \in \mathbb{F}$. And at this time, $D = ad(-kL_0 + e_0 H_0)$. $\qquad\square$

# 3. The automorphism group of $\mathcal{L}$

Denote by $Aut\mathcal{L}$ the automorphism group. By the relations of $\mathcal{L}$, we have that $G_\alpha^+, G_\beta^-$ are locally ad-nilpotent elements for any $\alpha, \beta \in \Gamma$. In this section, we denote $\mathbb{F}^*$ the multiply group which is generated by the non-zero elements of $\mathbb{F}$.

**Lemma 3.1.** For any $\sigma \in Aut\mathcal{L}$, $\sigma(G_\gamma^\pm) \in \text{span}_\mathbb{F}\{G_\alpha^+ \mid \alpha \in \Gamma\} \cup \text{span}_\mathbb{F}\{G_\alpha^- \mid \alpha \in \Gamma\}$.

**Proof.** Note that for any $\sigma \in Aut\mathcal{L}$, $\sigma(\mathcal{L}_{\bar{i}}) = \mathcal{L}_{\bar{i}}$, $\bar{i} \in \mathbb{Z}_2$, and $\sigma(x)$ is also a locally ad-nilpotent element if $x$ is a locally ad-nilpotent element, and $G_\alpha^+$ and $G_\beta^-$ are not abelian, then it is not difficult to prove that $\sigma(G_\gamma^\pm) \in \text{span}_\mathbb{F}\{G_\alpha^+ \mid \alpha \in \Gamma\} \cup \text{span}_\mathbb{F}\{G_\alpha^- \mid \alpha \in \Gamma\}$. $\qquad\square$

**Lemma 3.2.** For any $\sigma \in Aut\mathcal{L}$, we have that:

(1) $\sigma(L_0), \sigma(H_0) \in \text{span}_\mathbb{F}\{L_0, H_0, c\}$.

(2) $\sigma(G_\gamma^+) \in \text{span}_\mathbb{F}\{G_\alpha^+ \mid \alpha \in \Gamma\}$ or $\text{span}_\mathbb{F}\{G_\alpha^- \mid \alpha \in \Gamma\}$ for all $\gamma \in \Gamma$.

**Proof.** (1) It is easy to see that $L_0, H_0, c, G_\alpha^\pm$, for all $\alpha \in \Gamma$, are locally finite elements of $\mathcal{L}$. And by lemma 3.1, we can prove (1).

(2) If there exist $\gamma_1, \gamma_2 \in \Gamma$, satisfy that $\sigma(G_{\gamma_1}^+) \in \text{span}_\mathbb{F}\{G_\alpha^+ \mid \alpha \in \Gamma\}$, $\sigma(G_{\gamma_2}^+) \in \text{span}_\mathbb{F}\{G_\alpha^- \mid \alpha \in \Gamma\}$, then we have that

$$\sigma([H_{\gamma_2 - \gamma_1}, G_{\gamma_1}^+]) = [\sigma(H_{\gamma_2 - \gamma_1}), \sigma(G_{\gamma_1}^+)] \in \text{span}_\mathbb{F}\{G_\alpha^+ \mid \alpha \in \mathbb{F}\},$$

and

$$\sigma([H_{\gamma_2 - \gamma_1}, G_{\gamma_1}^+]) = \sigma(G_{\gamma_2}^+) \in \text{span}_\mathbb{F}\{G_\alpha^- \mid \alpha \in \mathbb{F}\},$$

it is a contradiction. $\qquad\square$

**Proposition 3.3.** For any $\sigma \in Aut\mathcal{L}$, there exist $f \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{F}^*), \xi, \varepsilon \in \{\pm 1\}, a \in \Gamma, b \in \mathbb{F}^*$, such that

$$\begin{aligned}
\sigma(L_\alpha) &= \varepsilon f(\alpha) L_{\varepsilon\alpha} + f(\alpha) a H_{\varepsilon a} + \tfrac{1}{6}\varepsilon a^2 \delta_{\alpha,0} c, \\
\sigma(H_\alpha) &= \xi(f(\alpha) H_{\varepsilon\alpha} + \tfrac{a}{3}\varepsilon\delta_{\alpha,0} c), \\
\sigma(G_\alpha^+) &= f(\alpha) b(\delta_{\xi,1} G_{\varepsilon(\alpha+a)}^+ + \delta_{\xi,-1} G_{\varepsilon(\alpha-a)}^-), \\
\sigma(G_\alpha^-) &= \varepsilon f(\alpha) b^{-1}(\delta_{\xi,-1} G_{\varepsilon(\alpha+a)}^+ + \delta_{\xi,1} G_{\varepsilon(\alpha-a)}^-), \\
\sigma(c) &= \varepsilon c,
\end{aligned} \qquad (3.1)$$

where $\delta_{m,n}$ is also the Kronecker notation.

**Proof.** By lemma 3.1 and 3.2, we can assume that

$$\sigma(L_0) = a_0 L_0 + b_0 H_0 + c_0 c, \quad \sigma(H_0) = d_0 L_0 + e_0 H_0 + f_0 c,$$

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and

$$\sigma(L_\gamma) = \sum_{\alpha \in \Gamma} a_{\gamma,\alpha} L_\alpha + \sum_{\alpha \in \Gamma} b_{\gamma,\alpha} H_\alpha + c_\gamma c,$$

$$\sigma(H_\gamma) = \sum_{\alpha \in \Gamma} d_{\gamma,\alpha} L_\alpha + \sum_{\alpha \in \Gamma} e_{\gamma,\alpha} H_\alpha + f_\gamma c,$$

$$\sigma(c) = m_0 c,$$

where $a_0, b_0, c_0, d_0, e_0, f_0, m_0, a_{\gamma,\alpha}, b_{\gamma,\alpha}, c_\gamma, d_{\gamma,\alpha}, e_{\gamma,\alpha}, f_\gamma \in \mathbb{F}$.

**Claim.** $a_0 \neq 0$.

If $a_0 = 0$, then $\sigma(L_0) = b_0 H_0 + c_0 c$. Set $x \in \mathcal{L}_\alpha$, $\alpha \neq 0$, we have that $-\alpha \sigma(x) = [\sigma(L_0), \sigma(x)] = [b_0 H_0 + c_0 c, \sigma(x)] = 0$, it is impossible. Therefore $a_0 \neq 0$.

Acting $\sigma$ on $[L_0, L_\gamma] = -\gamma L_\gamma$, we have that

$$-a_0 \alpha \sum_{\alpha \in \Gamma} (a_{\gamma,\alpha} L_\alpha + b_{\gamma,\alpha} H_\alpha) = -\gamma \sum_{\alpha \in \Gamma} (a_{\gamma,\alpha} L_\alpha + b_{\gamma,\alpha} H_\alpha + c_\gamma c),$$

i.e.,

$$\begin{cases} (\gamma - a_0 \alpha) a_{\gamma,\alpha} = 0, \\ (\gamma - a_0 \alpha) b_{\gamma,\alpha} = 0, \\ c_\gamma = 0. \end{cases} \tag{3.2}$$

If there exists $\gamma_0 \in \Gamma$, such that $\dfrac{\gamma_0}{a_0} \notin \Gamma$, i.e., $\gamma_0 - a_0 \alpha \neq 0$ for all $\alpha \in \Gamma$, then $a_{\gamma_0,\alpha} = b_{\gamma_0,\alpha} = 0$ for all $\alpha \in \Gamma$, that is to say, $\sigma(L_{\gamma_0}) = 0$, it is a contradiction since $\sigma$ is an automorphism. Therefore, $\dfrac{\gamma}{a_0} \in \Gamma$ for all $\gamma \in \Gamma$, then $a_0 = \pm 1$, where 1 is the unit of $\mathbb{F}$.

**Case 1.** $a_0 = 1$.

By (3.2), we have that $\sigma(L_\gamma) = a_\gamma L_\gamma + b_\gamma H_\gamma + c_0 \delta_{\gamma,0} c$. And by the same argument, we also have that

$$\sigma(H_\gamma) = d_\gamma L_\gamma + e_\gamma H_\gamma + f_0 \delta_{\gamma,0} c.$$

Applying $\sigma$ on $[H_\alpha, H_\beta] = \frac{\alpha}{3} c \delta_{\alpha+\beta,0}$, we can obtain that

$$\frac{\alpha}{3} m_0 c \delta_{\alpha+\beta,0} = (\alpha-\beta) d_\alpha d_\beta L_{\alpha+\beta} + (\alpha d_\beta e_\alpha - \beta d_\alpha e_\beta) H_{\alpha+\beta} + (\frac{\alpha^3 - \alpha}{12} d_\alpha d_\beta + \frac{\alpha}{3} e_\alpha e_\beta) c \delta_{\alpha+\beta,0},$$

i.e.,

$$\begin{cases} (\alpha - \beta) d_\alpha d_\beta = 0, \\ \alpha d_\beta e_\alpha - \beta d_\alpha e_\beta = 0, \\ \frac{\alpha}{3} m_0 = \frac{\alpha^3 - \alpha}{12} d_\alpha d_{-\alpha} + \frac{\alpha}{3} e_\alpha e_{-\alpha}. \end{cases}$$

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If there exists $\alpha_0 \in \Gamma$, such that $d_{\alpha_0} \neq 0$, then we will deduce that $d_\beta = e_\beta = 0$ for all $\beta \in \Gamma \setminus \{\alpha_0, 0\}$, contradiction with $\sigma \in Aut\mathcal{L}$, hence $d_\alpha = 0$ for all $\alpha \in \Gamma$, and $\alpha m_0 = \alpha e_\alpha e_{-\alpha}$, i.e.,

$$d(H_\alpha) = e_\alpha H_\alpha + f_0 \delta_{\alpha,0} c.$$

By lemma 3.2, we can discuss the action of $\sigma$ on $\mathcal{L}_{\bar{1}}$ in two cases.

**Subcase 1.1.** $\sigma(G_\gamma^+) = \sum_{\alpha \in \gamma} g_{\gamma,\alpha} G_\alpha^+$.

Since $\sigma$ is an automorphism, by lemma 3.2, we have that $\sigma(G_\gamma^-) = \sum_{\alpha \in \Gamma} h_{\gamma,\alpha} G_\alpha^-$. By $\sigma[L_0, G_\gamma^+] = [\sigma(L_0), \sigma(G_\gamma^+)]$, we get that

$$(\gamma + b_0 - \alpha) g_{\gamma,\alpha} = 0,$$

therefore, $b_0 \in \Gamma$, and

$$\sigma(G_\gamma^+) = g_\gamma G_{\gamma+b_0}^+.$$

Similarly,

$$\sigma(G_\gamma^-) = h_\gamma G_{\gamma-b_0}^-.$$

Applying $\sigma$ to $[H_\alpha, G_\beta^\pm]$, we have that

$$\begin{cases} g_{\beta+\gamma} = e_\beta g_\gamma, \\ h_{\beta+\gamma} = e_\beta h_\gamma, \end{cases} \quad \text{for all } \beta, \gamma \in \mathbb{F}.$$

Then we can deduce that

$$g_\alpha = e_\alpha g_0, \qquad h_\alpha = e_\alpha h_0, \qquad e_0 = 1. \tag{3.3}$$

Applying $\sigma$ to $[G_\alpha^-, G_\beta^+]$, we have that

$$a_{\alpha+\beta} = g_\alpha h_\beta, \tag{3.4}$$

$$2b_{\alpha+\beta} - (\alpha - \beta)e_{\alpha+\beta} = -(\alpha - \beta - 2b_0)g_\alpha h_\beta, \tag{3.5}$$

$$\frac{1}{3}((\alpha^2 - b_0)^2 - \frac{1}{4})g_\alpha h_{-\alpha} = \frac{1}{3}(\alpha^2 - \frac{1}{4})m_0 - 2\alpha f_0 + 2c_0. \tag{3.6}$$

By (3.3) and (3.4),

$$g_0 h_0 = a_0 = 1,$$

and

$$a_{\alpha+\beta} = e_\alpha g_0 \cdot e_\beta h_0 = e_\alpha e_\beta,$$

then $a_\alpha = e_\alpha$ for all $\alpha \in \Gamma$. Furthermore, following (3.5), we obtain that

$$b_\alpha = b_0 e_\alpha = b_0 a_\alpha.$$

By (3.6), let $\beta = -\alpha$, one can get that

$$m_0 = 1, \qquad f_0 = \frac{1}{3}b_0, \qquad c_0 = \frac{1}{6}b_0^2.$$

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Therefore,

$$
\begin{aligned}
\sigma(L_\alpha) &= a_\alpha L_\alpha + a_\alpha b_0 H_\alpha + \tfrac{1}{6}b_0^2 c\delta_{\alpha,0}, & \sigma(H_\alpha) &= a_\alpha H_\alpha + \tfrac{1}{3}b_0 c\delta_{\alpha,0},\\
\sigma(G_\alpha^+) &= a_\alpha g_0 G_{\alpha+b_0}^+, & \sigma(G_\alpha^-) &= a_\alpha g_0^{-1} G_{\alpha-b_0}^-, & (3.7)\\
\sigma(c) &= c,
\end{aligned}
$$

where $a_\alpha \in \mathbb{F}^*$ satisfies that $a_{\alpha+\beta} = a_\alpha a_\beta$, $b_0 \in \Gamma$, $g_0 \in \mathbb{F}^*$.

**Subcase 1.2.** $\sigma(G_\gamma^+) = \sum_{\alpha\in\gamma} u_{\gamma,\alpha} G_\alpha^+$.

Under the same argument, we can obtain that

$$
\begin{aligned}
\sigma(L_\alpha) &= a_\alpha L_\alpha + a_\alpha b_0 H_\alpha + \tfrac{1}{6}b_0^2 c\delta_{\alpha,0}, & \sigma(H_\alpha) &= -a_\alpha H_\alpha - \tfrac{1}{3}b_0 c\delta_{\alpha,0},\\
\sigma(G_\alpha^+) &= a_\alpha u_0 G_{\alpha-b_0}^-, & \sigma(G_\alpha^-) &= a_\alpha u_0^{-1} G_{\alpha+b_0}^+, & (3.8)\\
\sigma(c) &= c,
\end{aligned}
$$

where $a_\alpha \in \mathbb{F}^*$ satisfies that $a_{\alpha+\beta} = a_\alpha a_\beta$, $b_0 \in \Gamma$, $u_0 \in \mathbb{F}^*$.

**Case 2.** $a_0 = -1$.

In this case, we have that $\sigma(L_0) = -L_0 + b_0 H_0 + c_0 c$. By the similar method, we can obtain the two following subcases:

**Subcase 2.1.**

$$
\begin{aligned}
\sigma(L_\alpha) &= -a_\alpha L_{-\alpha} + a_\alpha b_0 H_{-\alpha} - \tfrac{1}{6}b_0^2 c\delta_{\alpha,0}, & \sigma(H_\alpha) &= a_\alpha H_{-\alpha} - \tfrac{1}{3}b_0 c\delta_{\alpha,0},\\
\sigma(G_\alpha^+) &= a_\alpha g_0 G_{-\alpha-b_0}^+, & \sigma(G_\alpha^-) &= -a_\alpha g_0^{-1} G_{-\alpha+b_0}^-, & (3.9)\\
\sigma(c) &= -c.
\end{aligned}
$$

**Subcase 2.2.**

$$
\begin{aligned}
\sigma(L_\alpha) &= -a_\alpha L_{-\alpha} + a_\alpha b_0 H_{-\alpha} - \tfrac{1}{6}b_0^2 c\delta_{\alpha,0}, & \sigma(H_\alpha) &= -a_\alpha H_\alpha + \tfrac{1}{3}b_0 c\delta_{\alpha,0},\\
\sigma(G_\alpha^+) &= a_\alpha u_0 G_{-\alpha+b_0}^-, & \sigma(G_\alpha^-) &= -a_\alpha u_0^{-1} G_{-\alpha-b_0}^+, & (3.10)\\
\sigma(c) &= -c,
\end{aligned}
$$

where $a_\alpha \in \mathbb{F}^*$ satisfies that $a_{\alpha+\beta} = a_\alpha a_\beta$, $b_0 \in \Gamma$, $g_0, u_0 \in \mathbb{F}^*$. $\qquad\square$

By proposition 3.3, we can obtain that

$$
Aut\mathcal{L} = \{\sigma(f,\xi,\varepsilon,a,b) \mid f(\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{F}^*), \xi, \varepsilon \in \{\pm 1\}, a \in \Gamma, b \in \mathbb{F}^*\},
$$

where $\sigma(f,\xi,\varepsilon,a,b) \in Aut\mathcal{L}$ satisfies the following relations:

$$
\begin{aligned}
\sigma(f,\xi,\varepsilon,a,b)(L_\alpha) &= \varepsilon f(\alpha) L_{\varepsilon\alpha} + f(\alpha)a H_{\varepsilon\alpha} + \tfrac{1}{6}\varepsilon a^2 c\delta_{\alpha,0},\\
\sigma(f,\xi,\varepsilon,a,b)(H_\alpha) &= \xi(f(\alpha)H_{\varepsilon\alpha} + \tfrac{1}{3}\varepsilon ac\delta_{\alpha,0}),\\
\sigma(f,\xi,\varepsilon,a,b)(G_\alpha^+) &= f(\alpha)b(\delta_{\xi,1}G_{\varepsilon(\alpha+a)}^+ + \delta_{\xi,-1}G_{\varepsilon(\alpha-a)}^-), & (3.11)\\
\sigma(f,\xi,\varepsilon,a,b)(G_\alpha^-) &= \varepsilon f(\alpha)b^{-1}(\delta_{\xi,-1}G_{\varepsilon(\alpha+a)}^+ + \delta_{\xi,1}G_{\varepsilon(\alpha-a)}^-),\\
\sigma(f,\xi,\varepsilon,a,b)(c) &= \varepsilon c,
\end{aligned}
$$

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where $f(\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F}^*)$, i.e., $f(\alpha + \beta) = f(\alpha)f(\beta)$ for all $\alpha, \beta \in \Gamma$, and $\xi, \varepsilon \in \{\pm 1\}$, $a \in \Gamma, b \in \mathbb{F}^*$. Obviously, $\sigma(f, 1, 1, a, b)$, $\sigma(f, -1, 1, a, b)$, $\sigma(f, 1, -1, a, b))$ and $\sigma(f, -1, -1, a, b)$ are the automorphisms which are defined in (3.7)-(3.10). And $\sigma(f_1, \xi_1, \varepsilon_1, a_1, b_1) = \sigma(f_2, \xi_2, \varepsilon_2, a_2, b_2)$ if and only if $f_1 = f_2$, $\xi_1 = \xi_2, \varepsilon_1 = \varepsilon_2$, $a_1 = a_2, b_1 = b_2$.

$$\sigma(f_1, \xi_1, \varepsilon_1, a_1, b_1)\sigma(f_2, \xi_2, \varepsilon_2, a_2, b_2)$$
$$= \sigma(\varphi, \xi_1\xi_2, \varepsilon_1\varepsilon_2, \xi_1\xi_2\varepsilon_1 a_1 + \xi_2 a_2, f_1(\xi_2\varepsilon_2 a_2)b_1^{\xi_2}b_2),$$

where $\varphi$ satisfies that $\varphi(\alpha) = f_1(\varepsilon_2\alpha)f_2(\alpha)$ for any $\alpha \in \mathbb{F}$. Obviously, $Aut\mathcal{L}$ is not a ablian group. For any $f \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, we define $f^{-1}$ by $f^{-1}(\alpha) = (f(\alpha))^{-1}$ for any $\alpha \in \Gamma$. It is easy to see that $f^{-1}$ is also a homomorphism from $\Gamma$ to $\mathbb{F}^*$. Then for any $\sigma(f, \xi, \varepsilon, a, b) \in Aut\mathcal{L}$, we can get that

$$\sigma^{-1}(f, \xi, \varepsilon, a, b) = \sigma(f^{-\varepsilon}, \xi, \varepsilon, -\xi\varepsilon a, sgn(\xi + \varepsilon)f(a)b^{-1}),$$

where $sgn(\eta) = \begin{cases} 1, & \eta \geqslant 0 \\ -1, & \eta < 0 \end{cases}$, and $f^1 = f$.

Set $\tau = < \sigma(f, 1, 1, a, b) >$, it is not difficult to prove that $\tau$ is a normal subgroup of $Aut\mathcal{L}$. Then we can obtain the following main theorem:

**Theorem 3.4.** (1) $Aut\mathcal{L}$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F}^*) \times \mathbb{Z}'_2 \times \mathbb{Z}'_2 \times \Gamma \times \mathbb{F}^*$, where $\mathbb{Z}'_2 = \{\pm 1\}$, the group multiplication is given by:

$$(f_1, \xi_1, \varepsilon_1, a_1, b_1) \cdot (f_2, \xi_2, \varepsilon_2, a_2, b_2) = (\varphi, \xi_1\xi_2, \varepsilon_1\varepsilon_2, \xi_1\xi_2\varepsilon_1 a_1 + \xi_2 a_2, f_1(\xi_2\varepsilon_2 a_2)b_1^{\xi_2}b_2),$$

where $\varphi$ satisfies that $\varphi(\alpha) = f_1(\varepsilon_2\alpha)f_2(\alpha)$ for any $\alpha \in \mathbb{F}$.

(2) $Aut\mathcal{L}/\tau$ is a Klein group.

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